MATH 504 HOMEWORK 1

Due Wednesday, September 12.

Problem 1. *Let y be a set of ordinals.*

(1) *If y is nonempty, show that the $\in$-minimal element in y is unique.*
(2) *Show that $\bigcup y$ is an ordinal.*

For the following assume that $\alpha, \beta, \gamma, \delta, \xi$ are ordinals.

Problem 2. *Show that $\alpha < \beta$ implies that $\gamma + \alpha < \gamma + \beta$ and $\alpha + \gamma \leq \beta + \gamma$. Give an example to show that $\leq$ cannot be replaced with $<$. Also, show that*

$$\alpha \leq \beta \to (\exists! \delta)(\alpha + \delta = \beta).$$

Problem 3. *Show that if $\gamma > 0$, then $\alpha < \beta$ implies that $\gamma \cdot \alpha < \gamma \cdot \beta$ and $\alpha \cdot \gamma \leq \beta \cdot \gamma$. Give an example to show that $\leq$ cannot be replaced with $<$. Also, show that*

$$(\alpha \leq \beta \wedge \alpha > 0) \to (\exists! \delta, \xi)(\xi < \alpha \wedge \alpha \cdot \delta + \xi = \beta).$$

Problem 4. *Verify that ordinal exponentiation satisfies $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ and $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.*

Problem 5. *Show in ZF^- (i.e. the ZF axioms minus Foundation) that for any set X the following are equivalent:*
(a) X can be well ordered,
(b) There is a $C : (\mathcal{P}(X) \setminus \{0\}) \to X$ such that $\forall Y \subset X (Y \neq \emptyset \to C(Y) \in Y)$.

Hint for (b)$\to$ (a): Fix $p \neq X$, and let $C(Y) = p$ if $Y \notin \mathcal{P}(X) \setminus \{0\}$. Define by transfinite recursion,

$$F(\alpha) = C(X \setminus \{F(\xi) \mid \xi < \alpha\}).$$

Problem 6. *(Well founded recursion) Suppose that E is a well founded relation on a set P, and that $F : V \to V$ is some class function. Show that there exists a function G with domain P, such that for every $x \in P$, $G(x) = F(x, G(\{y \in P \mid yEx\}))$.*